# Damage Detection of Structural Systems with Noisy Incomplete Input and Response Measurements

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*ABSTRACT. A probabilistic approach for damage detection is presented using noisy incomplete input and response measurements that is an extension of a Bayesian system identification approach developed by the authors. This situation may be encountered, for example, during low-level ambient vibrations when a structure is instrumented with accelerometers that measure the input ground motion and structural response at a few locations but the wind excitation is not measured. A substructuring approach is used for the parameterization of the mass and stiffness distributions. Damage is defined to be a reduction of the substructure stiffness parameters compared with those of the undamaged structure. By using the proposed probabilistic methodology, the probability of various damage levels in each substructure can be calculated based on the available data. A four-story benchmark building subjected to wind and ground shaking is considered in order to demonstrate the proposed approach.*

## 1 INTRODUCTION

Recent interest has been shown in using Bayesian probabilistic approaches for model updating and damage detection as they allow for an explicit treatment of the uncertainties involved [1, 2, 3, 4, 5]. In [1], a methodology for model updating based on a Bayesian probabilistic system identification framework was presented. Although the framework is general, the presentation is for the case where the prediction error due to measurement noise and modeling error is modeled by Gaussian white noise. In the present paper, the prediction error is modeled as the sum of a filtered white noise process, representing the input measurement noise filtered through the system, plus another independent white noise process, representing the response measurement noise and modeling error. A Bayesian time-domain approach [2] is extended to handle the case of incomplete input measurements with measurement noises in both input and output measurements. The proposed approach allows for the direct calculation of the probability density function (PDF) of the model parameters based on the data. By using data from the initial undamaged state and a later possibly damaged state, the probability of damage in specified substructures may be calculated. An example using noisy simulated data from an ASCE benchmark problem is given for illustration.

## 2 MODEL FORMULATION

Consider a system with $N_d$ degrees-of-freedom (DOF) and equation of motion:

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{T}\mathbf{g} \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the mass, damping and stiffness matrix of the system, respectively, $\mathbf{g} = [\mathbf{g}_o^T \quad \mathbf{g}_u^T]^T \in \mathbb{R}^{N_g}$ is the actual forcing vector, which is comprised of $\mathbf{g}_o \in \mathbb{R}^{N_{go}}$ (the observed part) and $\mathbf{g}_u \in \mathbb{R}^{N_{gu}}$ (the unobserved part). Here, $\mathbf{g}_u$ is modeled as a vector of Gaussian i.i.d. variables with zero mean and covariance matrix $\mathbf{\Sigma}_{gu}(\boldsymbol{\theta}_n)$, where $\boldsymbol{\theta}_n$ is the parameter vector which characterizes the properties of the excitation and measurement noise. Also, $\mathbf{T} = [\mathbf{T}_o \quad \mathbf{T}_u] \in \mathbb{R}^{N_d \times N_g}$ is the forcing distribution matrix, where $N_g = N_{go} + N_{gu}$, so that $\mathbf{Tg} = \mathbf{T}_o\mathbf{g}_o + \mathbf{T}_u\mathbf{g}_u$. The mass and stiffness matrices, $\mathbf{M}$ and $\mathbf{K}$, are defined in terms of mass and stiffness parameters $\boldsymbol{\theta}_m$ and $\boldsymbol{\theta}_s$ by: $\mathbf{M} = \sum_{j=1}^{N_{sub}} \mathbf{M}_j(\boldsymbol{\theta}_m)$ and $\mathbf{K} = \sum_{j=1}^{N_{sub}} \mathbf{K}_j(\boldsymbol{\theta}_s)$, where $N_{sub}$ is the number of substructures and $\mathbf{M}_j$ and $\mathbf{K}_j$ are the contributions to the mass and stiffness matrix of the $j^{th}$ substructure, respectively. Furthermore, the damping matrix $\mathbf{C}$ is defined in terms of the modal damping ratios for classical normal modes.

We assume that discrete-time response data are available at $N_o$ observed DOFs where the measured response $\mathbf{z}(k) \in \mathbb{R}^{N_o}$ is a linear combination of the model state vector $\mathbf{y}(k) = [\mathbf{x}(k)^T \quad \dot{\mathbf{x}}(k)^T]^T$ and the actual force $\mathbf{g}(k)$, plus a prediction error term $\mathbf{n}_z(k) \in \mathbb{R}^{N_o}$, which accounts for modeling error and measurement noise in the response measurements. The index $k$ refers to time $(k-1)\Delta t$ where $\Delta t$ is the sampling interval. We model $\mathbf{n}_z$ as a zero-mean Gaussian i.i.d. vector process with covariance matrix $\mathbf{\Sigma}_{nz}(\boldsymbol{\theta}_n)$. We also assume that incomplete excitation measurements are available, given by the actual observed excitation $\mathbf{g}_o$ plus measurement noise $\mathbf{n}_f$, i.e. $\mathbf{f}(k) = \mathbf{g}_o(k) + \mathbf{n}_f(k)$, where $\mathbf{n}_f$ is assumed to be a Gaussian i.i.d. vector process with zero mean and covariance matrix $\mathbf{\Sigma}_{nf}(\boldsymbol{\theta}_n)$. Furthermore, we model $\mathbf{n}_z$, $\mathbf{n}_f$ and $\mathbf{g}_u$ as mutually independent. Thus,

$$\mathbf{z}(k) = \mathbf{L}_1\mathbf{y}(k)+\mathbf{L}_2\mathbf{Tg}(k)+\mathbf{n}_z(k) = \mathbf{L}_1\mathbf{y}(k)+\mathbf{L}_2\mathbf{T}_o\mathbf{f}(k)-\mathbf{L}_2\mathbf{T}_o\mathbf{n}_f(k)+\mathbf{L}_2\mathbf{T}_u\mathbf{g}_u(k)+\mathbf{n}_z(k) \quad (2)$$

where $\mathbf{L}_1 \in \mathbb{R}^{N_o \times 2N_d}$ and $\mathbf{L}_2 \in \mathbb{R}^{N_o \times N_d}$ are observation matrices, which depend on the type of measurements (e.g. displacements or accelerations), and $\mathbf{y}(k)$ is given by (1).

Suppose that $N_m$ modes contribute significantly to the response. Using modal analysis and considering only the first $N_m$ significant modes, one can obtain $N_m$ uncoupled modal equations of motion for the modal coordinates $\mathbf{q}(t)$ and modal forces $\mathbf{h}(t)$ using the following transformation:

$$\mathbf{x}(t) \approx \mathbf{\Phi}\mathbf{q}(t) \quad \text{and} \quad \mathbf{h}(t) = \mathbf{\Phi}^T\mathbf{Tg}(t) \quad (3)$$

where $\mathbf{\Phi} \in \mathbb{R}^{N_d \times N_m}$ is the modeshape matrix for the first $N_m$ modes, comprised of the modeshape vectors $\boldsymbol{\phi}^{(r)}$, $r = 1, \dots, N_m$, which are assumed to be mass normalized.

The parameter vector $\mathbf{a}$ for identification from the excitation and response data is comprised of: 1) the mass and stiffness parameters $\boldsymbol{\theta}_m$ and $\boldsymbol{\theta}_s$ that characterizes the mass matrix $\mathbf{M}$ and stiffness matrix $\mathbf{K}$; 2) damping ratios of the lowest $N_m$ modes: $\zeta_r, r = 1, ..., N_m$; 3) the elements of the $N_m \times N_{go}$ modal participation matrix $\mathbf{\Phi}^T\mathbf{T}_o$ (note that the $(r, s)$ element of the aforementioned matrix represents the $r^{th}$ modal participation factor corresponding to the $s^{th}$ observed input excitation); 4) the excitation and noise parameter vector $\boldsymbol{\theta}_n$; and 5) the initial conditions of the modal coordinates for the $N_m$ considered modes; thus, a total number of $2N_m$ initial conditions are to be identified. Often, in practice the system may be assumed to start from rest. In such a case, the initial conditions can be treated as constant and equal to zero and can be excluded from the vector $\mathbf{a}$ of parameters for identification.

Let $\mathbf{Z}_{m,p}$ and $\mathbf{F}_{m,p}$ denote the response and the excitation measurement matrix from time $(m-1)\Delta t$ to $(p-1)\Delta t$ ($m \leq p$), respectively:

$$\mathbf{Z}_{m,p} = [\mathbf{z}(m) \quad \cdots \quad \mathbf{z}(p)] \quad \text{and} \quad \mathbf{F}_{m,p} = [\mathbf{f}(m) \quad \cdots \quad \mathbf{f}(p)], \quad m \leq p \quad (4)$$

Let $N$ denote the total number of points in time where measurements are available. The approach to damage detection is to first use the Bayesian framework presented in the next section to obtain the updated PDF $p(\mathbf{a}|\mathbf{Z}_{1,N}, \mathbf{F}_{1,N})$ of the parameter vector $\mathbf{a}$ given the measured input data $\mathbf{F}_{1,N}$ and output data $\mathbf{Z}_{1,N}$. Then this is used to compute the probability of damage in the $i^{th}$ substructure exceeding damage level d which is defined by

$$P_i^{dam}(d) = P\{\theta_i^{pd} < (1-d)\theta_i^{ud} | \mathbf{F}_{1,N}^{ud}, \mathbf{Z}_{1,N}^{ud}, \mathbf{F}_{1,N}^{pd}, \mathbf{Z}_{1,N}^{pd}\} \tag{5}$$

where subscripts 'ud' and 'pd' refer to undamaged and possibly damaged cases. This probability is evaluated from the updated marginal PDFs on the stiffness parameters as in [3] by using an asymptotic expansion for the integral involved [6].

## 3 BAYESIAN MODEL UPDATING

### 3.1 Exact Formulation

Using Bayes' theorem, the expression for the updated PDF of the parameters $\mathbf{a}$ given the measured response $\mathbf{Z}_{1,N}$ and the measured input $\mathbf{F}_{1,N}$ is:

$$p(\mathbf{a}|\mathbf{Z}_{1,N}, \mathbf{F}_{1,N}) = c_1 p(\mathbf{a}) p(\mathbf{Z}_{1,N}|\mathbf{a}, \mathbf{F}_{1,N}) \tag{6}$$

where $c_1$ is a normalizing constant such that the integral of the right hand side of (6) over the domain of $\mathbf{a}$ is equal to unity. The factor $p(\mathbf{a})$ in (6) denotes the prior PDF of the parameters and it may be chosen based on engineering judgement. It may be treated as constant (noninformative prior) if all values of the parameters over some large but finite domain are felt to be equally plausible a priori. The factor $p(\mathbf{Z}_{1,N}|\mathbf{a}, \mathbf{F}_{1,N})$ is the dominant factor on the right hand side of (6). It reflects the contribution of the measured data $\mathbf{Z}_{1,N}$ and $\mathbf{F}_{1,N}$ in establishing the updated (posterior) PDF of $\mathbf{a}$, $p(\mathbf{a}|\mathbf{Z}_{1,N}, \mathbf{F}_{1,N})$, which gives a measure of the relative plausibility between any two values of $\mathbf{a}$. The latter depends only on the relative values of the prior PDF $p(\mathbf{a})$ and the relative values of $p(\mathbf{Z}_{1,N}|\mathbf{a}, \mathbf{F}_{1,N})$. In order to establish the most probable (plausible) value of $\mathbf{a}$, denoted by $\hat{\mathbf{a}}$ and referred to as the optimal parameters, one therefore needs to maximize $p(\mathbf{a})p(\mathbf{Z}_{1,N}|\mathbf{a}, \mathbf{F}_{1,N})$.

Since we are considering linear systems and both the measurement noise and unmeasured excitation are assumed to be Gaussian, it follows that the factor $p(\mathbf{Z}_{1,N}|\mathbf{a}, \mathbf{F}_{1,N})$ is an $N_o N$-variate Gaussian distribution with appropriately calculated mean and covariance matrix. Direct calculation of this factor for different values of $\mathbf{a}$ becomes computationally prohibitive for large number $N$ of data, as it requires repeated calculation of the determinant and inverse of the corresponding very high dimensional $N_o N \times N_o N$ covariance matrices. Thus, although (6) offers a theoretically exact solution to the modal updating problem, its computational implementation poses a challenge. In the next section, we present an approximation which overcomes this difficulty and renders the Bayesian model updating problem computationally feasible.

### 3.2 Proposed Approximation

The PDF $p(\mathbf{Z}_{1,N}|\mathbf{a},\mathbf{F}_{1,N})$ in (6) can be written as a product of conditional probabilities:

$$\begin{aligned} p(\mathbf{Z}_{1,N}|\mathbf{a},\mathbf{F}_{1,N}) =& p(\mathbf{Z}_{1,N_p}|\mathbf{a},\mathbf{F}_{1,N}) \prod_{k=N_p+1}^{N} p(\mathbf{z}(k)|\mathbf{a},\mathbf{Z}_{1,k-1},\mathbf{F}_{1,N}) \\ \approx& p(\mathbf{Z}_{1,N_p}|\mathbf{a},\mathbf{F}_{1,N}) \prod_{k=N_p+1}^{N} p(\mathbf{z}(k)|\mathbf{a},\mathbf{Z}_{k-N_p,k-1},\mathbf{F}_{1,N}) \end{aligned} \tag{7}$$

Here, each conditional probability factor depending on all the previous response measurements is approximated by a conditional probability depending on only the most recent $N_p$ response measurements. The sense of this approximation is that response measurements too far in the past do not provide significant information about the present observed response. Of course, one expects this to be true if $N_p$ is so large that all the correlation functions have decayed to very small values. However, it will be shown with a numerical example later in Section 4.1 that a significantly smaller value of $N_p$ suffices for the approximation in (7) to be valid for practical purposes. In particular, it is found that a value for $N_p$ of the order of $\frac{T_0}{\Delta t}$ is sufficient, where $T_0$ is the fundamental period of the system and $\Delta t$ is the sampling interval. For example, assuming $\Delta t = \frac{1}{25}T_0$, it follows that a value of $N_p = 25$ is sufficient. The advantage of the approximation in (7) will become obvious in the subsequent sections where the expressions for computing the factors on the right hand side of (7) are given. In Section 3.2.1, the expression for the joint probability distribution $p(\mathbf{Z}_{1,N_p}|\mathbf{a},\mathbf{F}_{1,N})$ in (7) is given. In Section 3.2.2, a general expression for the conditional probability $p(\mathbf{z}(k)|\mathbf{a},\mathbf{Z}_{k-N_p,k-1},\mathbf{F}_{1,N})$ in (7) is derived. Based on these results, $p(\mathbf{Z}_{1,N}|\mathbf{a},\mathbf{F}_{1,N})$ can be computed efficiently from (7).

The optimal (most probable) parameters $\hat{\mathbf{a}}$ can then be obtained by minimizing $u(\mathbf{a}) = -\ln[p(\mathbf{a})\, p(\mathbf{Z}_{1,N}|\mathbf{a},\mathbf{F}_{1,N})]$. It is found that the updated PDF of the parameters $\mathbf{a}$ can be well approximated by a Gaussian distribution $N(\hat{\mathbf{a}},\mathbf{H}^{-1}(\hat{\mathbf{a}}))$ with mean $\hat{\mathbf{a}}$ and covariance matrix $\mathbf{H}^{-1}(\hat{\mathbf{a}})$, where $\mathbf{H}(\hat{\mathbf{a}})$ denotes the Hessian of $u(\mathbf{a})$ calculated at $\mathbf{a} = \hat{\mathbf{a}}$.

#### 3.2.1 Expression for $p(\mathbf{Z}_{1,N_p}|\mathbf{a},\mathbf{F}_{1,N})$ in (7)

Since linear systems are considered, and both the unmeasured excitation and measurement noise are Gaussian, the joint probability distribution $p(\mathbf{Z}_{1,N_p}|\mathbf{a},\mathbf{F}_{1,N})$ follows an $N_oN_p$-variate Gaussian distribution. The expressions for the mean and covariance of this distribution are derived as a function of the identification parameter vector $\mathbf{a}$ as follows.

Let $\mathbf{y}_q$ denote the modal state vector, i.e. $\mathbf{y}_q = [\mathbf{q}^T \quad \dot{\mathbf{q}}^T]^T$. In analogy to (3), one can write:

$$\mathbf{y} \approx \bar{\mathbf{\Phi}}\mathbf{y}_q \tag{8}$$

where $\bar{\mathbf{\Phi}}$ is given by:

$$\bar{\mathbf{\Phi}} = \begin{bmatrix} \mathbf{\Phi} & \mathbf{0}_{N_d\times N_m} \\ \mathbf{0}_{N_d\times N_m} & \mathbf{\Phi} \end{bmatrix} \tag{9}$$

Also, from (2) and (8), one obtains: $\mathbf{z}(k) \approx \mathbf{L}_1\bar{\mathbf{\Phi}}\mathbf{y}_q(k) + \mathbf{L}_2\mathbf{T}\mathbf{g}(k) + \mathbf{n}_z(k)$.

Next, consider the system equation for the modal state vector $\mathbf{y}_q$:

$$\dot{\mathbf{y}}_q = \mathbf{A}\mathbf{y}_q + \mathbf{E}\mathbf{\Phi}^T\mathbf{T}\mathbf{g} \tag{10}$$

where $\mathbf{A} = \begin{bmatrix} \mathbf{0}_{N_m} & \mathbf{I}_{N_m} \\ -diag(\omega_1^2, \ldots, \omega_{N_m}^2) & -2diag(\zeta_1\omega_1, \ldots, \zeta_{N_m}\omega_{N_m}) \end{bmatrix}$ and $\mathbf{E} = \begin{bmatrix} \mathbf{0}_{N_m} \\ \mathbf{I}_{N_m} \end{bmatrix}$.

The continuous-time differential equation (10) yields the following difference equation:

$$\mathbf{y}_q(k+1) = \mathbf{A}_d\mathbf{y}_q(k) + \mathbf{E}_d\mathbf{\Phi}^T\mathbf{T}\mathbf{g}(k) \tag{11}$$

where $\mathbf{y}_q(k)$ denotes the modal state vector at time $t = (k-1)\Delta t$, $\mathbf{A}_d \equiv e^{\mathbf{A}\Delta t}$ and $\mathbf{E}_d \equiv \mathbf{A}^{-1}(\mathbf{A}_d - \mathbf{I}_{2N_m})\mathbf{E}$. For notational convenience, denote the relationship between the modal state vector and the input of the above system using the function $\mathcal{L}_q$: $\mathbf{y}_q(k) \equiv \mathcal{L}_q(k; \mathbf{a}, \mathbf{\Phi}^T\mathbf{T}\mathbf{G}_{1,N}), k \leq N$, where $\mathbf{G}_{1,N}$ denotes, in analogy to the definition of (4), the matrix comprised of the actual input force time history up to time $(N-1)\Delta t$, i.e. $\mathbf{G}_{1,N} = [\mathbf{g}(1) \quad \mathbf{g}(2) \quad \cdots \quad \mathbf{g}(N)]$. It can be easily shown that the mean $\boldsymbol{\mu}(k) \equiv E[\mathbf{z}(k)|\mathbf{a}, \mathbf{F}_{1,N}]$ is given by:

$$\boldsymbol{\mu}(k) = (\mathbf{L}_1\bar{\mathbf{\Phi}})\mathcal{L}_q(k; \mathbf{a}, \mathbf{T}_o\mathbf{F}_{1,N}) + \mathbf{L}_2\mathbf{T}_o\mathbf{f}(k) \tag{12}$$

Thus, $\boldsymbol{\mu}(k)$ is equal to the model response calculated assuming that the only input is the measured excitation. The difference between $\mathbf{z}(k)$ and $\boldsymbol{\mu}(k)$ is the prediction error $\boldsymbol{\upsilon}(k)$:

$$\boldsymbol{\upsilon}(k) = \mathbf{z}(k) - \boldsymbol{\mu}(k) \tag{13}$$

It is worth noting that $\boldsymbol{\mu}(k)$ in (12) can be simply calculated using the function 'lsim' in MATLAB. Collecting all the terms calculated by (12), $E(\mathbf{Z}_{1,N_p}|\mathbf{a}, \mathbf{F}_{1,N})$ is given by:

$$E(\mathbf{Z}_{1,N_p}|\mathbf{a}, \mathbf{F}_{1,N}) = [\boldsymbol{\mu}^T(1) \quad \cdots \quad \boldsymbol{\mu}^T(N_p)]^T \tag{14}$$

The covariance matrix $\mathbf{\Sigma}_{Z,N_p} \equiv E\left[[\mathbf{Z}_{1,N_p} - E(\mathbf{Z}_{1,N_p}|\mathbf{a}, \mathbf{F}_{1,N})][\mathbf{Z}_{1,N_p} - E(\mathbf{Z}_{1,N_p}|\mathbf{a}, \mathbf{F}_{1,N})]^T\right]$ is given by:

$$\mathbf{\Sigma}_{Z,N_p} = \begin{bmatrix} \mathbf{\Sigma}_\upsilon(1,1) & \cdots & \mathbf{\Sigma}_\upsilon(1,N_p) \\ & \ddots & \vdots \\ sym & & \mathbf{\Sigma}_\upsilon(N_p,N_p) \end{bmatrix} \tag{15}$$

where $\mathbf{\Sigma}_\upsilon(m,p)$ can be approximated by:

$$\begin{aligned} \mathbf{\Sigma}_\upsilon(m,p) \approx \mathbf{\Sigma}_\upsilon^\star(p-m) = &(\mathbf{L}_1\bar{\mathbf{\Phi}})\mathbf{V}\mathbf{B}\mathbf{V}^T(\mathbf{A}_d^T)^{p-m}(\mathbf{L}_1\bar{\mathbf{\Phi}})^T + (\mathbf{L}_2\mathbf{T}_o\mathbf{\Sigma}_{nf}\mathbf{T}_o^T\mathbf{L}_2^T + \mathbf{\Sigma}_{nz})\delta_{m,p} \\ &+ \mathbf{L}_2\mathbf{\Sigma}_{nf}\mathbf{T}_o^T\mathbf{\Phi}\mathbf{E}_d^T(\mathbf{A}_d^T)^{p-m-1}(\mathbf{L}_1\bar{\mathbf{\Phi}})^T(1-\delta_{m,p}) \end{aligned} \tag{16}$$

This can be obtained by following the same procedure as in the Appendix in [2]. Here, $\mathbf{B}$ is given by:

$$B^{(j,l)} = \frac{1-(\lambda_j\lambda_l)^{N-1}}{1-\lambda_j\lambda_l}(\mathbf{V}^{-1}\mathbf{E}_d\mathbf{\Sigma}_\tau\mathbf{E}_d^T\mathbf{V}^{-T})^{(j,l)} \tag{17}$$

where $\lambda_j$ and $\mathbf{V}$ are the $j^{th}$ eigenvalue and eigenvector matrix of $\mathbf{A}_d$, respectively; and $\mathbf{\Sigma}_\tau \equiv \mathbf{\Phi}^T(\mathbf{T}_o\mathbf{\Sigma}_{nf}\mathbf{T}_o^T + \mathbf{T}_u\mathbf{\Sigma}_{gu}\mathbf{T}_u^T)\mathbf{\Phi}$.

The joint probability distribution $p(\mathbf{Z}_{1,N_p}|\mathbf{a}, \mathbf{F}_{1,N})$ follows an $N_oN_p$-variate Gaussian distribution:

$$p(\mathbf{Z}_{1,N_p}|\mathbf{a}, \mathbf{F}_{1,N}) = \frac{|\mathbf{\Sigma}_{Z,N_p}|^{-\frac{1}{2}}}{(2\pi)^{\frac{N_oN_p}{2}}}\exp\left\{-\frac{1}{2}[\mathbf{Z}_{1,N_p} - E(\mathbf{Z}_{1,N_p}|\mathbf{a}, \mathbf{F}_{1,N})]^T\mathbf{\Sigma}_{Z,N_p}^{-1}[\mathbf{Z}_{1,N_p} - E(\mathbf{Z}_{1,N_p}|\mathbf{a}, \mathbf{F}_{1,N})]\right\} \tag{18}$$

### 3.2.2 Expression for $p(\mathbf{z}(k)|\mathbf{a}, \mathbf{Z}_{k-N_p,k-1}, \mathbf{F}_{1,N})$ in (7)

Define the vector $\mathbf{W}(k)$, $k > N_p$, as follows: $\mathbf{W}(k) = [\mathbf{z}^T(k) \quad \mathbf{z}^T(k-1) \quad \cdots \quad \mathbf{z}^T(k-N_p)]^T$. Note that $\mathbf{W}(k)$ is comprised of all the response measurements appearing in $p(\mathbf{z}(k)|\mathbf{a}, \mathbf{Z}_{k-N_p,k-1}, \mathbf{F}_{1,N})$. Specifically, $\mathbf{W}(k)$ consists of $\mathbf{z}(k)$ followed by all vector elements of $\mathbf{Z}_{k-N_p,k-1}$ ordered in a descending time index order. Next, we derive expressions for the mean value and the covariance of the Gaussian joint PDF $p(\mathbf{W}(k)|\mathbf{a}, \mathbf{F}_{1,N})$. Clearly, the expected value of the vector $\mathbf{W}(k)$ given $\mathbf{a}$ and $\mathbf{F}_{1,N}$ is given by:

$$E[\mathbf{W}(k)|\mathbf{a}, \mathbf{F}_{1,N}] = [\boldsymbol{\mu}^T(k) \quad \boldsymbol{\mu}^T(k-1) \quad \cdots \quad \boldsymbol{\mu}^T(k-N_p)]^T \tag{19}$$

The covariance matrix $\boldsymbol{\Sigma}_W(k) = E\{[\mathbf{W}(k) - E(\mathbf{W}(k)|\mathbf{a}, \mathbf{F}_{1,N})][\mathbf{W}(k) - E(\mathbf{W}(k)|\mathbf{a}, \mathbf{F}_{1,N})]^T\}$ given $\mathbf{F}_{1,N}$ is easily shown to be:

$$\boldsymbol{\Sigma}_W(k) = \begin{bmatrix} \mathbf{Q}_{1,1}(k) & \cdots & \mathbf{Q}_{1,N_p+1}(k) \\ & \ddots & \vdots \\ sym & & \mathbf{Q}_{N_p+1,N_p+1}(k) \end{bmatrix} \tag{20}$$

where $\mathbf{Q}_{m,p}(k)$ is given by: $\mathbf{Q}_{m,p}(k) = \boldsymbol{\Sigma}_v(k-m+1, k-p+1), 1 \leq m \leq p \leq N_p+1$. It can be shown that the matrices $\boldsymbol{\Sigma}_v(k, k+r)$ tend to a limit value $\boldsymbol{\Sigma}_v^\star(r)$, given by (16), as $k \to \infty$. In practice, this convergence is found to be achieved very fast, i.e. for relatively small values of $k$. By using (16), we obtain the following approximation:

$$\boldsymbol{\Sigma}_W(k) \approx \begin{bmatrix} \boldsymbol{\Sigma}_v^\star(0) & \boldsymbol{\Sigma}_v^\star(-1) & \cdots & \boldsymbol{\Sigma}_v^\star(-N_p) \\ \boldsymbol{\Sigma}_v^\star(1) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \boldsymbol{\Sigma}_v^\star(-1) \\ \boldsymbol{\Sigma}_v^\star(N_p) & \cdots & \boldsymbol{\Sigma}_v^\star(1) & \boldsymbol{\Sigma}_v^\star(0) \end{bmatrix} \tag{21}$$

Therefore, the joint PDF $p(\mathbf{W}(k)|\mathbf{a}, \mathbf{F}_{1,N})$, $N_p+1 \leq k \leq N$, is approximated by an $N_o(N_p+1)$-variate Gaussian distribution with mean given by (19) and constant covariance matrix given by (21). Now, we partition the matrix $\boldsymbol{\Sigma}_W$ as follows:

$$\boldsymbol{\Sigma}_W = \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{12}^T & \boldsymbol{\Sigma}_{22} \end{bmatrix} \tag{22}$$

where $\boldsymbol{\Sigma}_{11}$, $\boldsymbol{\Sigma}_{12}$ and $\boldsymbol{\Sigma}_{22}$ have dimensions $N_o \times N_o$, $N_o \times N_oN_p$ and $N_oN_p \times N_oN_p$, respectively. Therefore, $\mathbf{e}(k) \equiv E[\mathbf{z}(k)|\mathbf{a}, \mathbf{Z}_{k-N_p,k-1}, \mathbf{F}_{1,N}]$ is given by:

$$\mathbf{e}(k) = \boldsymbol{\mu}(k) + \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}[\boldsymbol{v}^T(k-1) \quad \boldsymbol{v}^T(k-2) \quad \cdots \quad \boldsymbol{v}^T(k-N_p)]^T \tag{23}$$

where $\boldsymbol{v}(m)$, $m = k-N_p, \ldots, k-1$, are given by (13) and $\boldsymbol{\mu}(k)$ is given by (12).

The covariance matrix $\boldsymbol{\Sigma}_{\epsilon,N_p}(k)$ of the error $\boldsymbol{\epsilon}(k) = \mathbf{z}(k) - \mathbf{e}(k)$ is given by:

$$\boldsymbol{\Sigma}_{\epsilon,N_p}(k) \equiv E[\boldsymbol{\epsilon}(k)\boldsymbol{\epsilon}^T(k)] = \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{12}^T = \boldsymbol{\Sigma}_{\epsilon,N_p} \tag{24}$$

Thus, the conditional probability $p(\mathbf{z}(k)|\mathbf{a}, \mathbf{Z}_{k-N_p,k-1}, \mathbf{F}_{1,N})$ can approximated by the following Gaussian distribution:

$$p(\mathbf{z}(k)|\mathbf{a}, \mathbf{Z}_{k-N_p,k-1}, \mathbf{F}_{1,N}) \approx \frac{|\boldsymbol{\Sigma}_{\epsilon,N_p}|^{-\frac{1}{2}}}{(2\pi)^{\frac{N_o}{2}}} \exp\left\{-\frac{1}{2}[\mathbf{z}(k) - \mathbf{e}(k)]^T \boldsymbol{\Sigma}_{\epsilon,N_p}^{-1}[\mathbf{z}(k) - \mathbf{e}(k)]\right\} \tag{25}$$

where $\mathbf{e}(k)$ is given by (23) and $\mathbf{\Sigma}_{\epsilon,N_p}$ is given by (24).

The advantage of the approximation introduced in (7) is that all the conditional probabilities on the right hand side of (7) are conditional on exactly $N_p$ previous points and follow an $N_o$-variate Gaussian distribution with approximately the same covariance matrix $\mathbf{\Sigma}_{\epsilon,N_p}$ which, therefore, needs to be calculated only once for a given parameter set $\mathbf{a}$. Thus, to compute $p(\mathbf{Z}_{1,N}|\mathbf{a}, \mathbf{F}_{1,N})$ one needs to calculate the inverse and the determinant of only the matrices $\mathbf{\Sigma}_{Z,N_p}$, $\mathbf{\Sigma}_{22}$ and $\mathbf{\Sigma}_{\epsilon,N_p}$, of dimension $N_oN_p \times N_oN_p$, $N_oN_p \times N_oN_p$ and $N_o \times N_o$, respectively. This effort is much smaller than that required in an exact formulation where one needs to calculate the inverse and the determinant of a matrix of dimension $N_oN \times N_oN$, where $N >> N_p$ in general.

## 4 NUMERICAL EXAMPLE: Four-story Benchmark Structure

The strong direction of the ASCE four-story benchmark structure [7] is considered. Noisy displacements measurements, with $5\%$ rms noise, are assumed to be available at the $3^{rd}$ and $5^{th}$ floor over a time interval $T = 10\,\mathrm{sec}$, using a sampling interval $\Delta t = 0.004\,\mathrm{sec}$. The structure is assumed to be excited by a scaled version of the ground motion record from the 1940 El Centro earthquake and by ambient wind excitation. The earthquake ground motion is assumed to be measured with $10\%$ rms noise. However, the wind excitation is not assumed to be measured and it is modeled by a sequence of Gaussian i.i.d. variables, with variance $2.0kN^2$ for each DOF. The first four modal frequencies for the undamaged structure are $9.41$ Hz, $25.54$ Hz, $38.66$ Hz and $48.01$ Hz, respectively. The damping ratios are chosen to be $1.0\%$ for all modes. Damage detection using the proposed approach is based on changes between the following two cases:

Case 1: Undamaged structure (Damage Pattern 0 defined in [7]) with the wind excitation only.

Case 2: Only one brace in the first story has $1/3$ stiffness loss (Damage Pattern 6 defined in [7]) with both wind and earthquake excitation. Figure 1 shows the third floor displacement history and the contribution from the earthquake alone.

In both cases, $N_p = 25$ is used in (7) which corresponds to data points from just over one fundamental period of this building. Table 1 shows the identification results for the stiffness parameters for both cases. For each case, the second column in these tables corresponds to the actual values used for generation of the simulated measurement data; the third and fourth columns correspond to the identified optimal parameters and the corresponding standard deviations, respectively; and the last column lists the coefficient of variation (COV) for each parameter. Figure 2 shows the probability $P_i^{dam}(d_i)$ that damage in the $i^{th}$ story exceeds damage level $d_i$. It can be clearly seen that the first story has about $12\%$ damage but that there is probably no damage at other floors.

## 5 CONCLUDING REMARKS

A Bayesian approach to damage detection, location and assessment is presented using noisy incomplete excitation and response data. It is based on a Bayesian time-domain approach for updating the PDF of the model parameters of a linear MDOF system using dynamic data. The updated posterior PDF can be accurately approximated by a multi-variate Gaussian distribution where the calculated mean and covariance matrix offer an estimate of the optimal values of the model parameters and their associated uncertainties. The updated posterior PDFs from data in the undamaged state and in a possibly damaged state are used to calculate the probability of damage in each substructure. The approach was shown to successfully determine the location and probable level of damage from noisy measurement data.


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| Case | Parameter | Actual $\tilde{a}$ | Optimal $\hat{a}$ | S.D. $\sigma$ | COV $\alpha$ | Case | Parameter | Actual $\tilde{a}$ | Optimal $\hat{a}$ | S.D. $\sigma$ | COV $\alpha$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | $\theta_1$ | 1.0000 | 1.0063 | 0.0118 | 0.012 | 2 | $\theta_1$ | 0.8816 | 0.8826 | 0.0086 | 0.010 |
| | $\theta_2$ | 1.0000 | 0.9963 | 0.0150 | 0.015 | | $\theta_2$ | 1.0000 | 0.9948 | 0.0116 | 0.012 |
| | $\theta_3$ | 1.0000 | 1.0106 | 0.0131 | 0.013 | | $\theta_3$ | 1.0000 | 0.9958 | 0.0098 | 0.010 |
| | $\theta_4$ | 1.0000 | 1.0077 | 0.0135 | 0.014 | | $\theta_4$ | 1.0000 | 1.0083 | 0.0109 | 0.011 |

Table 1: Identification results of the stiffness parameters.

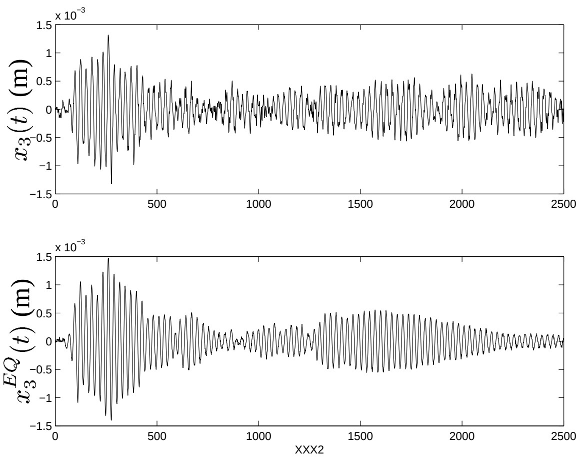


Figure 1: Response time histories at the third floor.

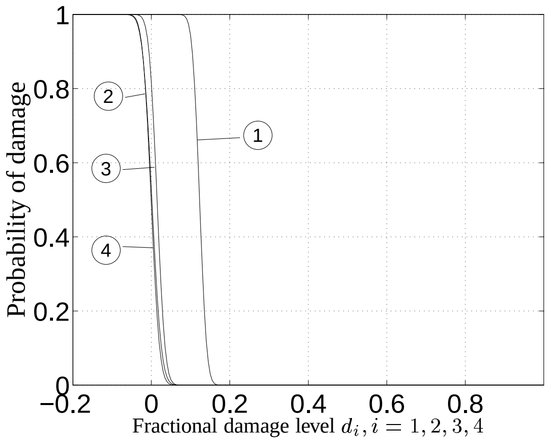


Figure 2: Probability of damage for the $i^{th}$ story stiffness loss exceeding $d_i$.